A Short and Elementary Proof of the Two-sidedness of the Matrix-Inverse

Pietro Paparella (pietrop@uw.edu) University of Washington Bothell

Beauregard [**2**] and Strang [**7**, pp. 88-89] give an elementary proof of the two-sidedness of the matrix-inverse using elementary matrices. Sandomierski [**6**] gives another elementary proof using an under-determined homogeneous linear system. Hill [**3**] provides a proof using basic results of rings and ideals, and Baksalary & Trenkler [**1**] provide a proof using the Moore-Penrose inverse.

However, as we shall see, the two-sidedness of the matrix inverse can be shown using only the notions of *linear independence* and the *reduced row-echelon form* of a matrix, concepts that feature prominently in an elementary linear algebra course (it is worth noting that the proof of Theorem 1 is in the same vein as that found in the advanced textbook by Meyer [**5**, Example 3.7.2]).

In addition, and as an application of this result, we prove that a matrix is invertible if and only if it is row-equivalent to the identity matrix without appealing to elementary matrices, which is common in textbooks (see, e.g., [**4, 7**]). This result is shown by utilizing the feasibility of certain linear systems and appealing to a well-known theorem.

Such demonstrations are valuable pedagogically, given that the linear system is the central object of study in a typical elementary linear algebra course.

The Proofs

In what follows all matrices are assumed to be n-by-n and all vectors are n-by-1. As is standard, the identity matrix is denoted by I.

Theorem 1. *If A and B are matrices such that $AB = I$, then $BA = I$.*

Proof. First, we show that the columns of B form a linearly independent set; to the contrary, if $Bx = 0$, where $x \neq 0$, then $x = Ix = ABx = A(Bx) = A(0) = 0$, a contradiction.

Because the columns of B form a linearly independent set, it follows that the reduced row-echelon form of B contains a pivot in every column and hence every row; following a well-known theorem [**4**, Theorem 4, §1.4], the equation $Bx = b$ has a solution for every vector b.

Next, we argue that the columns of A must form a linearly independent set; to that end, suppose $Ay = 0$, where $y \neq 0$. From previous, there is a necessarily nonzero vector x such that $y = Bx$. Thus, $0 = Ay = A(Bx) = (AB)x = Ix = x$, a contradiction.

Multiplying both sides of the equation $AB = I$ on the right by A yields $ABA = A$. Subtracting A from both sides and applying the distributive property yields $A(BA - I) = 0$.

We claim that the matrix $Z := BA - I = 0$; if not, then Z contains a nonzero entry and hence a nonzero column – say the j^{th}-column, which we denote by z_j. Thus, $Az_j = 0$, which is a contradiction unless $Z = BA - I = 0$, i.e., $BA = I$. ■

Theorem 2. *A matrix A is invertible if and only if A is row-equivalent to I.*

Proof. Following Theorem 1, A is invertible if and only if there is a matrix X such that $AX = I$. Thus, A is invertible if and only if the linear system $Ax = e_i$ is feasible for all $i \in \{1, \ldots, n\}$, where e_i denotes the i^{th} column of I.

We claim that if $Ax = e_i$ is feasible for all $i \in \{1, \ldots, n\}$, then $Ax = b$ is feasible for every vector b (note that the converse of this statement is obviously true). To that end, if s_i is any solution to $Ax = e_i$ and $b = [b_1 \ \cdots \ b_n]^\top$, then, following the linearity of A,

$$b = Ib := \sum_{i=1}^{n} b_i e_i = \sum_{i=1}^{n} b_i (A s_i) = \sum_{i=1}^{n} A(b_i s_i) = A\left(\sum_{i=1}^{n} b_i s_i\right),$$

and the claim is established.

Since $Ax = b$ for every vector b, another application of [**4**, Theorem 4, §1.4] reveals that A is invertible if and only if the reduced row-echelon form of A possesses a pivot in every row, i.e., A is invertible if and only if it is row-equivalent to I. ■

Remark. Our demonstration proves the equivalence of the following statements of the *invertible matrix theorem*:

- A is invertible.
- A is row-equivalent to I.
- $Ax = b$ is feasible for every vector b.
- the reduced row-echelon form of A has a pivot in every row.
- the columns of A are linearly independent.
- the equation $Ax = 0$ has only the trivial solution.
- the columns of A span $\mathbb{R}^n$.
- the columns of A form a basis for $\mathbb{R}^n$.

Summary. An elementary proof of the two-sidedness of the matrix-inverse is given using only linear independence and the reduced row-echelon form of a matrix. In addition, it is shown that a matrix is invertible if and only if it is row-equivalent to the identity matrix without appealing to elementary matrices. This proof underscores the importance of a *basis* and provides a proof of the *invertible matrix theorem*.

References

1. O. M. Baksalary and G. Trenkler, Another Proof of the Two-sidedness of Matrix Inverses, IMAGE, **50** (2013) 2.
2. R. A. Beauregard, A short proof of the two-sidedness of matrix inverses, *Math. Mag.* **80** (2) (2007) 135–136.
3. P. Hill, On the matrix equation $AB = I$, *American Math. Montly* **74** (7) (1967) 848–849. DOI: 10.2307/2315819
4. D. C. Lay, S. R. Lay, and J. J. McDonald, *Linear Algebra and Its Applications*, 5th ed., Pearson Education, 2016.
5. C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, PA, 2000.
6. F. Sandomierski, An elementary proof of the two-sidedness of matrix inverses, *Math. Mag.* **85** (4) (2012) 289. DOI: 10.4169/math.mag.85.4.289
7. W. G. Strang, *Introduction to Linear Algebra*, 5th ed., Wellesley-Cambridge, Wellesley, MA, 2016.

